A

H

1. 75% of the customers of ACME Mutual Insurance have auto insurance, and 40% have homeowners insurance. What is the maximum possible probability that a randomly selected customer with auto insurance does not have homeowners insurance?

(A) 20%

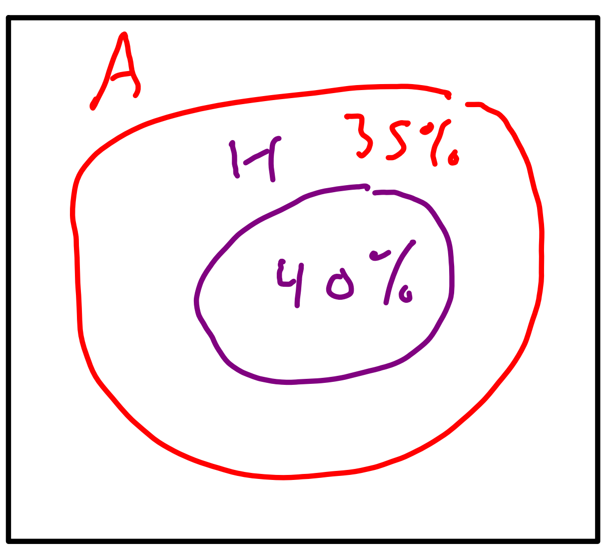
(B) 40%

(C) 60%

(D) 80%

(E) 100%

$$P[H^c \mid A] = \frac{P[H^c \text{ and } A]}{P[A]} = \frac{60\%}{75\%} = D$$



max overlap

min overlap

2. Suppose Z is a normal random variable with $\mathrm{E}X = 2$ and coefficient of variation 3. Find $\mathrm{P}[Z > 1]$.

(A) .05

(B) .34

(C) .57

(D) .66

(E) .95

Coeff of Variation

$$3 = \frac{\text{SD of } Z}{EZ} = \frac{\text{SD of } Z}{2}$$

$$6 = \text{SD of } Z$$

$$\frac{Z-2}{6} \text{ is a std normal}$$

$$P[Z>1] = P\left[\frac{Z-2}{6} > \frac{1-2}{6}\right]$$

$$= P\left[\text{std normal} > \frac{-1}{6}\right]$$

$$= 1 - \Phi\left(\frac{-1}{6}\right) = 1 - \left(1 - \Phi\left(\frac{1}{6}\right)\right)$$

$$= \Phi\left(\frac{1}{6}\right) \approx .57$$

3. The random variables X and Y are uniformly distributed over the set $X, Y > 0$ and $X + Y < 1$. What is the variance of X?

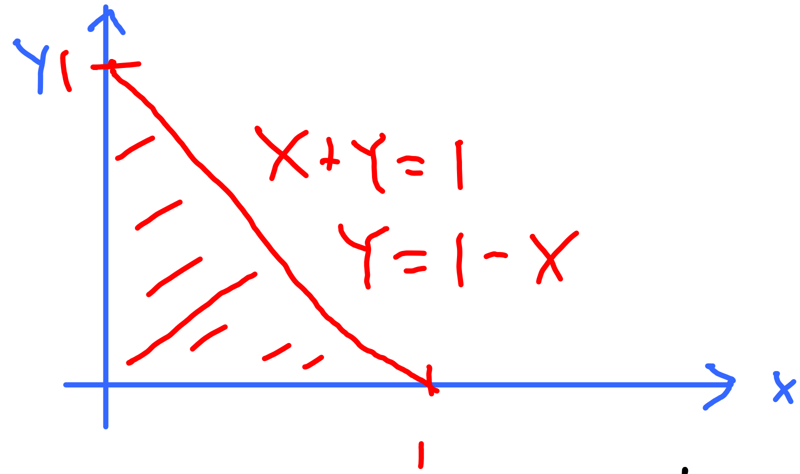
(A) $1/36$

(B) $2/36$

(C) $3/36$

(D) $4/36$

(E) $6/36$



$$\text{density of } (X,Y) = \frac{1}{\text{area}} = \frac{1}{\frac{1}{2}} = 2$$

$$EX = \int_0^1 \int_0^{1-x} 2 \cdot x \, dy \, dx = \int_0^1 2x(1-x)\, dx$$

$$= \int_0^1 2x - 2x^2 \, dx = \left(x^2 - \frac{2x^3}{3}\right)\Big|_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$$

$$EX^2 = \int_0^1 \int_0^{1-x} 2 \cdot x^2 \, dy \, dx = \int_0^1 2x^2(1-x)\, dx$$

$$= \int_0^1 2x^2 - 2x^3 \, dx = \frac{2}{3}x^3 - \frac{2}{4}x^4 \Big|_0^1 = \frac{2}{3} - \frac{1}{2}$$

$$= \frac{1}{6} \qquad \text{Var} X = \left(\frac{1}{6}\right) - \left(\frac{1}{3}\right)^2$$

$$= \frac{1}{6} - \frac{1}{9} = \frac{6-4}{36} = \frac{2}{36}$$

4. The probability that Roger Federer wins a tennis match in straight sets is 70%. Assuming that the outcome of each match is independent, what is the probability that in his next 7 matches that he will win in straight sets at least 5 times?

(A) .13

(B) .33

(C) .44

(D) .65

(E) .96

X = # of times wins in straight sets

$X \sim Binom(7, .7)$

Want $P[X \geq 5]$

$= P[X=5] + P[X=6] + P[X=7]$

$= \binom{7}{5} .7^5 .3^2 + \binom{7}{6} .7^6 .3^1 + .7^7$

$= \frac{7 \cdot 6}{2} (.7)^5 (.3)^2 + 7 (.7)^6 (.3)^1 + .7^7$

$= .318 + .247 + .082 = .65$

5. X and Y have joint density given by

$$f_{X,Y}(x,y) = \frac{3}{16}|x|y^3$$

for $-2 \le x \le 2$ and $0 \le y \le |x|$ and $f_{X,Y}(x,y) = 0$ otherwise. Find EY.

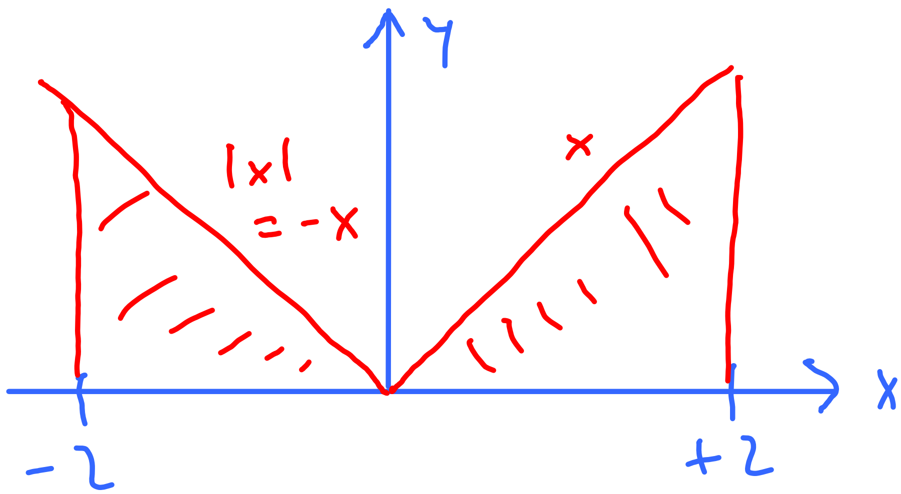
(A) 0.7

(B) 0.9

(C) 1.2

(D) 1.4

(E) 1.6



$$EY = 2\int_0^2\int_0^x y\,\frac{3}{16}\,x y^3\,dy\,dx$$

by symmetry

$$= \frac{3}{8}\int_0^2\int_0^x x y^4\,dy\,dx$$

$$= \frac{3}{8}\int_0^2 \frac{x y^5}{5}\Big|_0^x\,dx = \frac{3}{8}\int_0^2 \frac{x^6}{5}\,dx$$

$$= \frac{3}{8}\,\frac{x^7}{5\cdot 7}\Big|_0^2 = 1.4$$

6. The joint cdf of X, Y is given by

$F(x, y) = 1 - e^{-x} - (1 - e^{-x(y+1)})/(y + 1)$

for $x, y \geq 0$. Find $P[1 < X \leq 2, 1 < Y \leq 3]$

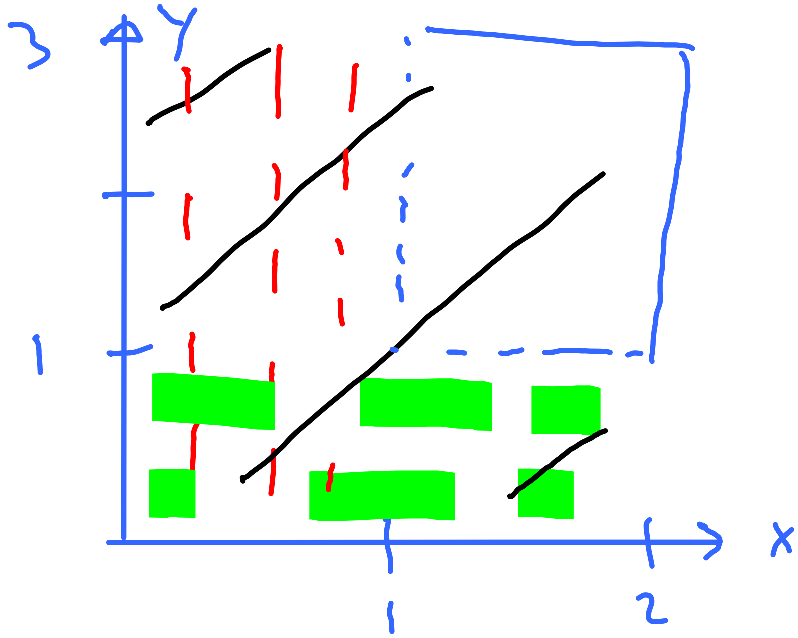
(A) .05

(B) .17

(C) .19

(D) .41

(E) .61



$F(2,3) = P[X \leq 2, Y \leq 3]$

$F(1,3) = P(X \leq 1, Y \leq 3)$

$F(2,1) = P[X \leq 2, Y \leq 1]$

$F(1,1) = P[X \leq 1, Y \leq 1]$

$P[1 < X \leq 2, 1 < Y \leq 3]$

$= F(2,3) - F(1,3) - F(2,1) + F(1,1)$

$= .05$

7. Let X be the number of rolls of a fair die before getting a 6, and let Y be the number of rolls before the first even number. Find $\mathrm{E}[X \mid Y = 5]$.

(A) 5

(B) 6

(C) 7

(D) 8

$\boxed{\text{(E) 9}}$

Method 1

$$E[X \mid Y=5] = \sum_x x\, P[X=x \mid Y=5]$$

$$= \sum_{n=1}^{\infty} P[X \geq n \mid Y=5]$$

Note: $X \geq Y = 5$

$P[X \geq n \mid Y=5] = 1$ if $n \leq 5$

$P[X \geq 6 \mid Y=5] = \frac{2}{3}$

$P[X \geq 7 \mid Y=5] = \frac{2}{3} \cdot \frac{5}{6}$

$P[X \geq 8 \mid Y=5] = \frac{2}{3} \cdot \left(\frac{5}{6}\right)^2$

$\vdots$

$$E[X \mid Y=5] = \underbrace{1+1+\dots+1}_{5 \text{ terms}} + \frac{2}{3} + \frac{2}{3}\cdot\frac{5}{6} + \frac{2}{3}\left(\frac{5}{6}\right)^2 \dots$$

$$= 5 + \frac{2/3}{1-5/6} = 5 + \frac{2/3}{1/6} = 9$$

7. Let X be the number of rolls of a fair die before getting a 6, and let Y be the number of rolls before the first even number. Find $\mathrm{E}[X \mid Y = 5]$.

(A) 5

(B) 6

(C) 7

(D) 8

$\boxed{\text{(E) 9}}$

Method 2

$$E[Z] = E[Z \mid \text{Case 1}]\, P[\text{Case 1}] + E[Z \mid \text{Case 2}] \cdot P[\text{Case 2}]$$

$$E[X \mid Y=5]$$
$$= E[X \mid Y=5, X=5]\, P[X=5 \mid Y=5] + E[X \mid Y=5, X>5]\, P[X>5 \mid Y=5]$$

$$P[X=5 \mid Y=5] = \frac{1}{3} \qquad P[X>5 \mid Y=5] = \frac{2}{3}$$

$$E[X \mid Y=5, X=5] = 5$$

$$E[X \mid Y=5, X>5] = 5 + EX$$
$$= 5 + 6$$

$$E[X \mid Y=5] = \frac{1}{3} \cdot 5 + \frac{2}{3}(5+6)$$
$$= 5 + \frac{2}{3} \cdot 6 = 9$$

8. Suppose X and Y are bivariate normal random variables with $\mathrm{E}X = \mathrm{E}Y = 1$ and $\mathrm{Var}X = \mathrm{Var}Y = 4$. If $\mathrm{Cov}(X,Y) = 3$, find $\mathrm{P}[2X - 3Y > 0]$.

(A) .40

(B) .45

(C) .53

(D) .55

(E) .60

$2X-3Y$ is a normal.

First, find mean & var.

$$E[2X-3Y] = 2EX - 3EY$$
$$= 2\cdot 1 - 3\cdot 1 = -1$$

$$\mathrm{Var}(2X-3Y) = 4\,\mathrm{Var}X - 2\cdot 2\cdot 3\cdot \mathrm{Cov}(X,Y) + 3^2\,\mathrm{Var}Y$$

$$= 4\cdot 4 - 12\cdot 3 + 9\cdot 4$$

$$= 16 \qquad SD = 4$$

$$P\left[\frac{2X-3Y--1}{4} > \frac{0--1}{4}\right] = 1-\Phi\left(\tfrac{1}{4}\right)$$

std normal

$$= 1 - .60 = .40$$

9. Suppose that I roll two independent dice, one red and one blue. Let A be the event that the blue die is even, B the event that the red die is even, and C the event that the sum is even. Which of the following is true?

~~(A)~~ None of them are independent

~~(B)~~ A and B are pairwise independent, but neither is pairwise independent of C

~~(C)~~ A and C are pairwise independent, as are B and C, but A and B are not pairwise independent

(D) All three pairs are pairwise independent, but it is not true that all three are mutually independent

~~(E)~~ All three are mutually independent

$$P[A] = \frac{1}{2} = P[B] = P[C]$$

$$P[AB] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4},$$ A & B are pairwise ind

$$P[AC] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$ as are A & C

$$P[ABC] = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \neq P[A]P[B]P[C]$$

so aren't all 3 mutually independent.

10. The moment generating function of X is given by $M_X(t) = 4/(4-t^2)$ for $-2 < t < 2$. What is the moment generating function of $1.2X$?

(A) $\dfrac{4}{4-1.2t^2}$

(B) $\dfrac{4}{4-1.44t^2}$

(C) $\dfrac{4.8}{4-t^2}$

(D) $\dfrac{4.8}{4-1.2t^2}$

(E) $\dfrac{5.76}{4-1.44t^2}$

$$M_X(t) = Ee^{tX}$$

$$M_{1.2X}(t) = Ee^{t(1.2X)}$$

$$= Ee^{(t \cdot 1.2)X}$$

$$= M_X(1.2t)$$

$$= \frac{4}{4-(1.2t)^2}$$

$$= \frac{4}{4-1.44t^2}$$

11. For $0 < x < y < z < 1$, the joint density of (X, Y, Z) is given by $f(x, y, z) = 48xyz$. Find $P[Y > 1/2]$.

(A) .75

(B) .78

(C) .81

(D) .84

(E) .87

$$\int_{1/2}^{1}\int_{y}^{1}\int_{0}^{y} f \, dx\, dz\, dy$$

$$= \int_{1/2}^{1}\int_{y}^{1}\int_{0}^{y} 48xyz \, dx\, dz\, dy$$

$$= \int_{1/2}^{1}\int_{y}^{1} 24x^2yz \Big|_{x=0}^{x=y} dz\, dy$$

$$= \int_{1/2}^{1}\int_{y}^{1} 24y^3z \, dz\, dy = \int_{1/2}^{1} 12y^3z^2 \Big|_{y}^{1}$$

$$= \int_{1/2}^{1} (12y^3 - 12y^5)\, dy = (3y^4 - 2y^6)\Big|_{1/2}^{1}$$

$$= .84$$

12. A student taking a 30 question multiple choice test and knows the answer to 24 of the questions. Whenever the student doesn't know the answer to a question, he chooses uniformly from one of the 5 choices. Given that the student gets a randomly chosen question right, what is the probability that the student guessed on the question?

(A) 1/25

(B) 1/21

(C) 1/18

(D) 1/11

(E) 1/5

$P[\text{guessed} \mid \text{answer correct}]$

$= \dfrac{P[\text{guess \& correct}]}{P[\text{correct}]}$

$= \dfrac{\frac{6}{30} \cdot \frac{1}{5}}{\frac{24}{30} \cdot 1 + \frac{6}{30} \cdot \frac{1}{5}}$

$= \dfrac{.04}{.84} = \dfrac{1}{21}$

13. Suppose $X > 0$ is a random variable with density xe^{-x}. Find the density for $Y = X^2$

(A) $2e^{-\sqrt{y}}$

(B) $2ye^{-\sqrt{y}}$

(C) $2\sqrt{y}e^{-\sqrt{y}}$

(D) $\frac{1}{2}e^{-\sqrt{y}}$

(E) $\sqrt{y}e^{-\sqrt{y}}$

Want

$$\frac{d}{dy} P[Y \le y]$$

$$= \frac{d}{dy} P[X \le \sqrt{y}]$$

$$= \frac{d}{dy} \int_0^{\sqrt{y}} xe^{-x}\,dx$$

$y = g(x)$

$$= \left(\sqrt{y}\, e^{-\sqrt{y}}\right) \frac{d}{dy}\left(\sqrt{y}\right)$$

$f_Y(y)$

$= f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$

$$= \cancel{\sqrt{y}}\, e^{-\sqrt{y}} \cdot \frac{1}{2\cancel{\sqrt{y}}}$$

$$= \frac{e^{-\sqrt{y}}}{2}$$

14. If X, Y, and Z are i.i.d. exponential random variables with mean 3, what is $\mathrm{E}[(X + Y + Z)^2]$?

(A) 27

(B) 54

(C) 81

(D) 108

(E) 135

$EX = 3$

$\mathrm{Var}\, X = 3^2$ b/c X is exponential

$EX^2 = 3^2 + 3^2 = 2 \cdot 3^2 = 18$

$E[(X+Y+Z)^2] = E[X^2 + Y^2 + Z^2 + 2XY + 2XZ + 2YZ]$

$= EX^2 + EY^2 + EZ^2 + 2(E[XY] + E(XZ) + \ldots)$

$= 3EX^2 + 2 \cdot 3\, EX\, EY$

$= 3 \cdot 18 + 2 \cdot 3 \cdot 3 \cdot 3$

$= 108$

15. Let U and V be independent, continuous uniform random variables on the interval $[1, 5]$. Find $\mathrm{P}[\min\{U, V\} < 2 \mid \max\{U, V\} > 2]$

(A) 3/8

(B) 2/5

(C) 3/5

(D) 5/8

(E) 2/3

$$\frac{P[\min < 2 \ \& \max > 2]}{P[\max > 2]}$$

$$P[\max > 2] = 1 - P[\max \le 2]$$

$$= 1 - P[U \le 2, V \le 2]$$

$$= 1 - \frac{1}{4} \cdot \frac{1}{4} = \frac{15}{16}$$

$$P[\min < 2, \max > 2] =$$

$$= P[U < 2, V > 2] + P[U > 2, V < 2]$$

$$= \frac{2-1}{5-1} \cdot \frac{5-2}{5-1} + \frac{3}{4} \cdot \frac{1}{4}$$

$$= 2 \cdot \frac{3}{4} \cdot \frac{1}{4}$$

so $P[\min < 2 \mid \max > 2] = \dfrac{\frac{2 \cdot 3}{16}}{\frac{15}{16}} = \dfrac{2}{5}$

16. Suppose that X and Y are independent, Poisson random variables with $\mathrm{E}X = 2$ and $\mathrm{E}Y = 2.5$. Find $\mathrm{P}[X + Y < 3]$

(A) .17

(B) .21

(C) .25

(D) .29

(E) .34

Key idea: independent Poisson random vars sum to a Poisson

$X+Y \sim \text{Poisson}(\underbrace{2+2.5}_{4.5})$

$$P[X+Y<3] = e^{-\lambda}\left[1+\lambda+\frac{\lambda^2}{2}\right]$$

$$= .0111 \times 15.625$$

$$= .17$$

17. Let N be a randomly chosen integer with $1 \leq N \leq 1,000$. What is the probability that N is not divisible by $7, 11$, or 13?

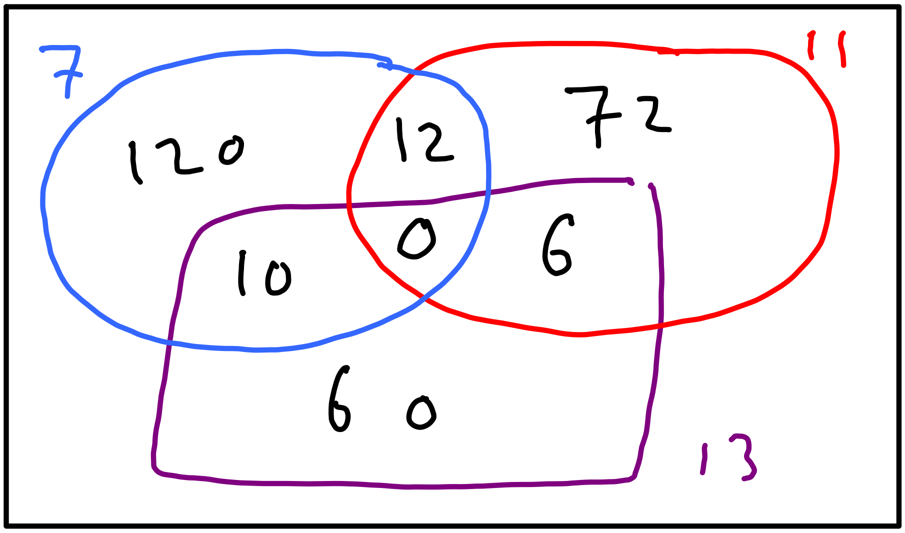
(A) .66

(B) .69

(C) .72

(D) .75

(E) .78



$7 \cdot 11 \cdot 13 = 1001 > 1000$, so no #s divisible by all 3

$\frac{1000}{7 \cdot 11} = 12.98$, so are 12 #s divisible by 7 & 11

$\left\lfloor \frac{1000}{7} \right\rfloor = 142 = 11 \cdot 13 - 1$ ~~30~~

$\frac{1000}{11} = 90.9$, so $\exists$ 90 #s divisible by 11

divisible by none $= 1000 - 120 - 12 - 10 - \ldots = 720$

18. Insurance losses L in a given year have a lognormal distribution with $L = e^X$, where X is a normal random variables with mean 3.9 and standard deviation 0.8. If a \$100 deductible and a \$50 benefit limit are imposed, what is the probability that the insurance company will pay the benefit limit given that a loss exceeds the deductible?

(A) .10

(B) .27

(C) .43

(D) .66

(E) .88

$$P[L \geq 150 \quad | L > 100]$$

$$= \frac{P[L \geq 150, \cancel{L > 100}]}{P[L > 100]}$$ redundant

$$= \frac{P[X \geq \ln 150]}{P[X > \ln 100]}$$

$$= \frac{P\left[\frac{X - 3.9}{.8} \geq \frac{\ln 150 - 3.9}{.8}\right]}{P[X > \ln 100]} = \frac{1 - \Phi(1.39)}{1 - \Phi(.88)}$$

$$= .43$$

19. A fair 6-sided die is rolled 1,000 times. Using a normal approximation with a continuity correction, what is the probability that the number of 3's that are rolled is greater than 150 and less than 180?

(A) .78

(B) .81

(C) .84

(D) .88

(E) .95

$X = \#$ of 3s

$EX = \frac{1{,}000}{6}$

$\mathrm{Var}\, X = 1000 \cdot \frac{1}{6} \cdot \frac{5}{6}$

N be normal, $\mu_N = \frac{1000}{6}$

$\sigma_N^2 = \frac{5000}{36}$

Want $P[150.5 \leq N < 179.5]$

$= P\left[\frac{150.5 - \mu_N}{\sigma_N} \leq \frac{N - \mu_N}{\sigma_N} \leq \frac{179.5 - \mu_N}{\sigma_N}\right]$

std normal

$= \Phi(1.09) - \Phi(-1.37)$

20. Four red dice and six blue dice are rolled. Assuming that all ten dice are fair six sided dice, and rolls are independent, what is the probability that exactly three of the red dice are even, and exactly two of the blue dice come up ones?

(A) .05

(B) .10

(C) .16

(D) .21

(E) .27

$$\underbrace{4 \cdot \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)}_{\text{red dice}} \cdot \underbrace{\frac{6 \cdot 5}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^4}_{\text{blue dice}}$$

$= .05$

21. A life insurance company classifies its customers as being either high risk or low risk. If 20% of the customers are high risk, and high risk customers are three times as likely as low risk customers to file a claim, what percentage of claims that are filed come from high risk customers?

(A) 30%

(B) 37%

(C) 43%

(D) 54%

(E) 60%

$$\frac{P[\text{High, claim}]}{P[\text{claim}]}$$

$$= \frac{\frac{1}{\cancel{5}} \cdot 3\cancel{p}}{\frac{1}{5} \cdot 3\cancel{p} + \frac{4}{\cancel{5}}\cancel{p}} = \frac{3}{7}$$

p = prob low risk files claim

22. Suppose that $X_1, \ldots X_{100}$ are random variables with $\mathrm{E}X_i = 100$ and $\mathrm{E}X_i^2 = 10,100$. If $\mathrm{Cov}(X_i, X_j) = -1$ for $i \neq j$, what is $\mathrm{Var}S$, where $S = \sum_{i=1}^{100} X_i$.

(A) 0

(B) 100

(C) 1,000

(D) 5,050

(E) 10,000

$$\mathrm{Var}\, S = \sum_{i=1}^{100} \mathrm{Var}\, X_i + \boxed{2 \sum_{i>j}} \mathrm{Cov}(X_i, X_j)$$

$$2 \sum_{i>j} = \sum_{i \neq j}$$

$$\mathrm{Var}\, X_i = EX_i^2 - (EX_i)^2$$

$$= 10,100 - 10,000 = 100$$

$$\mathrm{Var}\, S = 100 \cdot 100 + (-1)\binom{100}{2} \cdot 2$$

$$= 100 \cdot 100 - 100 \cdot 99$$

$$= 100(100 - 99)$$

$$= 100$$

23. The moment generating function of X is $M_X(t) = e^{2t^2-5t}$. Find Var$X$.

(A) 1

(B) 2

(C) 3

(D) 4

(E) 5

$$EX = \frac{d}{dt} M_X(t) \Big|_{t=0}$$

$$= \frac{d}{dt} e^{2t^2-5t} \Big|_{t=0}$$

$$= (4t-5) e^{2t^2-5t} \Big|_{t=0}$$

$$= -5e^0 = -5$$

$$EX^2 = \frac{d^2}{dt^2} M_X(t) \Big|_{t=0}$$

$$= \frac{d}{dt} \left[(4t-5) e^{2t^2-5t} \right]_{t=0}$$

$$= 4e^{(\quad)} + (4t-5)^2 e^{(\quad)} \Big|_{t=0}$$

$$= 4 + 25 = 29$$

$$\text{Var } X = 29 - (-5)^2 = 29 - 25$$

$$= 4$$

24. The cdf of a random variable X satisfies

$$F(X) = 1 - \frac{200^2}{(X+200)^2}$$

for $x > 0$. Find P[$50 < X < 300 \mid X > 100$].

(A) .22

(B) .36

(C) .51

(D) .64

(E) .78

$$\frac{P[50 < x < 300 \ \& \ x > 100]}{P[X > 100]}$$

$$= \frac{P[100 < x < 300]}{P[x > 100]}$$

$$= \frac{F(300) - F(100)}{1 - F(100)}$$

$$= \frac{1 - \left(\frac{200}{500}\right)^2 - \left(1 - \left(\frac{200}{300}\right)^2\right)}{\left(\frac{200}{100+200}\right)^2}$$

$$= .64$$

25. The density of Y is proportional to y^2 for $0 < y < 3$, and is 0 otherwise. Find the 80th percentile of Y.

(A) .9

(B) 1.3

(C) 1.8

(D) 2.3

(E) 2.8

$$f_Y(y) = cy^2$$

$$1 = \int_0^3 cy^2\,dy$$

$$1 = \frac{cy^3}{3}\Big|_0^3 = \frac{c \cdot 27}{3}$$

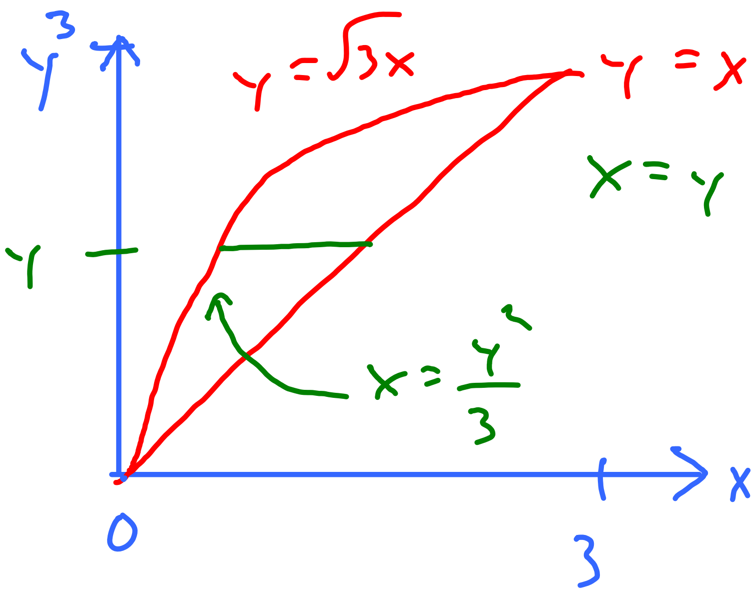
$$c = \frac{1}{9}$$

$$.8 = \int_0^t \frac{1}{9} y^2\,dy = \frac{t^3}{27}$$

$$(27)(.8) = t^3$$

$$2.8 = t$$

26. Suppose that (X, Y) is uniformly chosen from the set given by $0 < X < 3$ and $x < y < \sqrt{3x}$. Find the marginal density $f_Y(y)$ of Y.



(A) $\frac{2}{3}\left(y - \frac{y^2}{3}\right)$

(B) $y - \frac{y^2}{3}$

(C) $\frac{3}{2}\left(y - \frac{y^2}{3}\right)$

(D) $\frac{3}{2}\left(\sqrt{3x} - x\right)$

(E) $\frac{2}{3}\left(\sqrt{3x} - x\right)$

Step 1: find density

$= \frac{1}{\text{area}}$ b/c uniform

$= \int_0^3 \int_x^{\sqrt{3x}} 1 \, dy \, dx$

$= \int_0^3 (\sqrt{3x} - x)\, dx = \sqrt{3} \cdot x^{3/2} \cdot \frac{2}{3} - \frac{x^2}{2} \Big|_0^3$

$= 6 - 4.5 = \frac{3}{2}$

density $= \frac{2}{3}$

Step 2: integrate joint density

$\int_{y^2/3}^{y} \frac{2}{3}\, dx = \frac{2}{3}\left(y - \frac{y^2}{3}\right)$

27. If X is a Poisson random variable with $P[X = 1] = 2.5P[X = 0]$, then what is the probability that X will be within 1 standard deviation of EX?

(A) .08

(B) .29

(C) .47

(D) .63

(E) .81

$P[X=1] = 2.5\,P[X=0]$

$\lambda \cancel{e^{-\lambda}} = 2.5 \cancel{e^{-\lambda}}$

$\lambda = 2.5 =$ mean

$=$ variance

SD ≈ 1.58

$P[2.5 - 1.58 < X < 2.5 + 1.58]$

$= P[X = 1, 2, 3, 4]$

$= e^{-\lambda}\left(\lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} \right)$

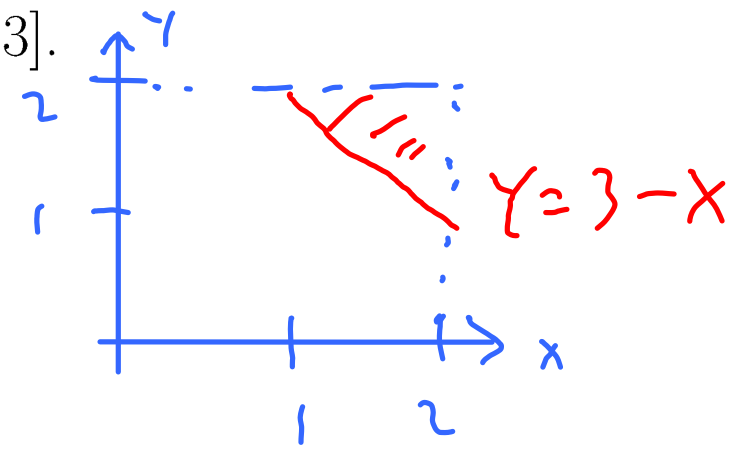
$= e^{-2.5}\left(2.5 + \frac{2.5^2}{2} + \frac{2.5^3}{6} + \frac{2.5^4}{24} \right)$

$= .809$

28. For $0 \le x, y \le 2$, the joint density of X and Y is

$$f_{X,Y}(x,y) = c(6 - x - 2y),$$

and the joint density is 0 otherwise. Find $P[X + Y \ge 3]$.



(A) 1/24

(B) 3/24

(C) 5/24

(D) 7/24

(E) 9/24

1°: find c

$$\int_0^2 \int_0^2 c(6-x-2y)\,dy\,dx = 1$$

$$c\int_0^2 (12-2x-4)\,dx = c(8\cdot 2 - 2^2)$$

$$= c\cdot 12 = 1, \quad c = \frac{1}{12}$$

$$P[X+Y\ge 3] = \int_1^2 \int_{3-x}^2 \frac{1}{12}(6-x-2y)\,dy\,dx$$

$$= \int_1^2 \left(\frac{6-x}{12}[2-3+x] - \frac{2^2-(3-x)^2}{12}\right)dx$$

$$\int_1^2 \left(\frac{6-x}{12} \left[\overbrace{2-3+x}^{-1+x} \right] - \frac{2^2-(3-x)^2}{12} \right) dx$$

$$= \frac{1}{12} \int_1^2 -6 + 7x - x^2 - 4 + 9 - 6x + x^2 \, dx$$

$$= \frac{1}{12} \int_1^2 (x-1) \, dx \qquad u = x-1$$

$$= \frac{1}{12} \int_0^1 u \, du = \frac{1}{24}$$

29. The joint density of X and Y is

$$f(x,y) = \frac{x^2 + 2xy}{10}$$

for $0 < x < 2$ and $x < y < x + 1$. Find $\mathrm{E}[Y \mid X = 1]$

(A) 1.1

(B) 1.3

(C) 1.5

(D) 1.7

(E) 1.9

$$f_{Y|X}(y \mid X=x) = \frac{f(x,y)}{\int_{\text{all } y} f(x,y)\,dy}$$

$$E[Y \mid X=x] = \int y \, f_{Y|X}(y \mid X=x)$$

$$\text{so } \frac{\int_1^{1+1} y \frac{1+2y}{\cancel{10}}\,dy}{\int_1^{1+1} \frac{1+2y}{\cancel{10}}\,dy} = \frac{\int_1^2 y + 2y^2\,dy}{\int_1^2 1 + 2y\,dy}$$

$$= \frac{\left.\frac{y^2}{2} + \frac{2y^3}{3}\right|_1^2}{\left. y + y^2 \right|_1^2} = \frac{\frac{4}{2} + \frac{16}{3} - \frac{1}{2} - \frac{2}{3}}{2 + 4 - 1 - 1}$$

$$= 1.54$$

30. Suppose that $X_1, \ldots, X_5$ are i.i.d., uniform random variables on the interval $(3, 6)$. Let $\bar{X}$ denote the average of X_1 through X_5, and let $\sigma_{\bar{X}}$ and $\mu_{\bar{X}}$ denote the standard deviation and mean of $\bar{X}$. Find the probability that the minimum and maximum of $X_1, \ldots X_5$ both differ from $\mu_{\bar{X}}$ by less that $\sigma_{\bar{X}}$.

$$EX_1 = \frac{3+6}{2}, \quad Var X_1 = \frac{(6-3)^2}{12}$$

(A) .00001

(B) .00032

(C) .00057

(D) .00083

(E) .00115

$$\mu_{\bar{x}} = E\,\frac{X_1 + \cdots + X_5}{5}$$

$$= \frac{1}{5}(5 \cdot EX_1) = EX_1 = 4.5$$

$$\sigma_{\bar{x}}^2 = Var\,\frac{X_1 + \cdots + X_5}{5} = \frac{1}{5^2}(5 \cdot Var X_1)$$

$$\sigma_{\bar{x}}^2 = \frac{1}{5} Var X_1 = \frac{1}{5} \cdot \frac{9}{12} = .15$$

$$P[4.5 - \sqrt{.15} < \min \,\&\, \max < 4.5 + \sqrt{.15}]$$

$$= P[4.5 - \sqrt{.15} < X_1, \ldots, X_5 < 4.5 + \sqrt{.15}]$$

$$= \left(\frac{2\sqrt{.15}}{3}\right)^5 = .00115$$

Answers

(1) D
(2) C
(3) B
(4) D
(5) D
(6) A
(7) E
(8) A
(9) D
(10) B
(11) D
(12) B
(13) D
(14) D
(15) B
(16) A
(17) C
(18) C
(19) A
(20) A
(21) C
(22) B
(23) D
(24) D
(25) E
(26) A
(27) E
(28) A
(29) C
(30) E