FAQ Q23: Jensen’s Inequality

Source Author: Paul Wilmott (2009)

Video By: Zak Fischer, FSA, CERA
Overview

- Convex Functions
- Jensen’s Inequality
- Importance in Finance
Convex Function Definition

Convexity

A function \( f \) is convex on an interval if for every \( x \) and \( y \) in that interval

\[
f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y)
\]

for any \( 0 \leq \lambda \leq 1 \).
Examples of Convex Functions

- Convex functions:
  - All linear functions
  - $y = x^2$, $y = x^4$
  - Payoff and profit functions for long calls/puts

- Not convex functions:
  - $y = \sqrt{x}$
  - $y = -x^2$
Jensen’s Inequality

If $f$ is a convex function and $x$ is a random variable then

$$E(f(x)) \geq f(E(x))$$

Example: $f(x) = x^2$, $E(X^2) \geq E(X)^2$

Also can write it as: $f(E(x)) \leq E(f(x))$
Approximating Convexity

- $E(f(x)) \geq f(E(x)) \Rightarrow E(f(S)) = f(E(S)) + \text{Convexity}$
- Let $\bar{S} = E(S)$ and $S = \bar{S} + \epsilon$
- $E(f(S))$
  $$= E(f(\bar{S} + \epsilon))$$
  $$= E \left[ f(\bar{S}) + \epsilon f'(\bar{S}) + \frac{1}{2} \epsilon^2 f''(\bar{S}) + \ldots \right]$$
  $$\approx E \left[ f(\bar{S}) + \epsilon f'(\bar{S}) + \frac{1}{2} \epsilon^2 f''(\bar{S}) \right]$$
  $$= f(\bar{S}) + E(\epsilon f'(\bar{S})) + E\left( \frac{1}{2} \epsilon^2 f''(\bar{S}) \right)$$
  $$= f(\bar{S}) + \frac{1}{2} E(\epsilon^2) f''(\bar{S})$$
  $$= f(E(S)) + \text{Convexity}$$

- Convexity $= \frac{1}{2} \cdot E(\epsilon^2) \cdot f''(\bar{S})$
  randomness function convexity
Jensen’s Inequality gives insight into why non-linear instruments such as options have inherent value.

Whenever a contract has convexity in a variable or parameter, and that variable is random, then an allowance must be made for this in pricing.
Let $X$ be the roll of a fair 6-sided die.

Compute $E(X^2)$ and $E(X)^2$.

Which is larger? Does Jensen’s inequality apply?
No peeking!
The expected roll is given by: 
\[ E(X) = \frac{1+2+3+4+5+6}{6} = 3.5 \]

The expected dollars received is given by: 
\[ E(X^2) = \frac{1^2+2^2+3^2+4^2+5^2+6^2}{6} = 15\frac{1}{6} \]

\( f(x) = x^2 \) is a convex function, so Jensen’s inequality applies.

Summarizing, we have that: 
\[ f(E(X)) = 3.5^2 = 12.25 < E(f(X)) = E(X^2) = 15\frac{1}{6} \]
Consider the following property:

\[ E(X^4) < E(X)^4 \]

Determine whether a random variable \( X \) exists such that the inequality above is satisfied.
No peeking!
Note that $f(x) = x^4$ is convex. Thus, by Jensen’s inequality, no such random variable exists, because Jensen’s inequality states that for a convex function:

$$E(f(x)) \geq f(E(x))$$

Thus,

$$E(X^4) \geq E(X)^4$$
Is \( f(x) = \sqrt{x} \) a convex function?
No peeking!
No, $f(x) = \sqrt{x}$ is not convex.

There are many ways to see this. For example, you could plug in the values below to see the convexity property is not satisfied. You could also draw a graph and see the secant line would be below the curve.

Let $x = 0$, $y = 2$, and $\lambda = .5$. Then,

$$f(\lambda x + (1 - \lambda)y) = f(1) = 1$$

$$f(x) + (1 - \lambda)f(y) = 0 + .5\sqrt{2} = .5\sqrt{2}$$

Thus,

$$f(\lambda x + (1 - \lambda)y) > \lambda f(x) + (1 - \lambda)f(y)$$

So $f(x) = \sqrt{x}$ is not convex.